RELATIVISTIC QUANTUM MECHANICS

Tuesday 06-05-2014, 18.30-21.30

On the first sheet write your name, address and student number. Write your name on **all** other sheets. You can earn up to six points for each subquestion; the total number of points is 90. Use conventions with $\hbar = c = 1$. The chiral representation of the 4×4 gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

PROBLEM 1: LORENTZ TRANSFORMATIONS

1.1 What are the generators of the Lorentz group for the vector representation? On how many components does this representation act? Indicate whether these are real or complex components.

1.2 What are the generators of the Lorentz group for the spinor representation? On how many components does this representation act? Indicate whether these are real or complex components.

1.3 Explain what the commutation relation of two generators have in common; in other words, why are these sets of matrices suitable as generators for the Lorentz group?

PROBLEM 2: HAMILTONIAN FORMALISM

The Lagrangian for a relativistic massive scalar field ϕ is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 \,. \tag{1}$$

2.1 What is the Euler-Lagrange equation for ϕ?

2.2 What is the momentum conjugate to ϕ, and what is the corresponding Hamiltonian density?

2.3 What are the Hamiltonian equations for this theory? Indicate the relations to the result of question 2.1.

PROBLEM 3: CAUSALITY

3.1 Explain the notion of causality in a classical theory in simple terms.

3.2 Does causality imply that the product of two quantum operators that are separated via a space-like trajectory vanishes? Does the Compton wavelength play any role here?

3.3 Does causality imply that the (anti-)commutator of two quantum operators that are separated via a space-like trajectory vanishes? Does the Compton wavelength play any role here?

PROBLEM 4: CANONICAL QUANTIZATION

The Lagrangian for a relativistic massive spinor field ψ is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \,. \tag{2}$$

4.1 What is the momentum Π conjugate to the field ψ?

4.2 What is the crucial difference between the field equations for bosonic and fermionic fields? What does this imply for the initial conditions that are needed to have a well-defined evolution of the field?

4.3 What is the Hamiltonian of this theory?

4.4 Which (anti-)commutation relations do we impose on the operators ψ and Π in the Schrödinger picture? Indicate the dependence on space and time.

The decomposition into plane waves and ladder operators $b_{\vec{p}}$ and $c_{\vec{p}}^\dagger$ reads

$$\psi = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} [b_{\vec{p}}^s u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s\,\dagger} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}}] \,. \tag{3}$$

4.5 What is the interpretation of the ladder operators? How many (anti-)particles does this correspond to?

4.6 What are the (anti-)commutation relations for the ladder operators?